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Upper bounds on Roman domination numbers of graphs[☆]

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ABSTRACT

A *Roman dominating function* of a graph G is a function $f: V(G) \to \{0, 1, 2\}$ such that whenever $f(v) = 0$ there exists a vertex u adjacent to v with $f(u) = 2$. The *weight* of f is $w(f) = \sum_{v\in V(G)} f(v)$. The *Roman domination number* $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function of G. This paper establishes a sharp upper bound on the Roman domination numbers of graphs with minimum degree at least 3. An upper bound on the Roman domination numbers of connected, big-claw-free and big-net-free graphs is also given.

1. Introduction

Roman domination is a variation of domination introduced by ReVelle [8,9]. Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighboring location only if there was a second army which would stay and protect the home. Thus, there are two types of armies, stationary and traveling. A vertex with no army must have a neighboring vertex with a traveling army. Stationary armies then dominate their own vertices. A vertex with two armies is dominated by its stationary army, and its open neighborhood is dominated by the traveling army. The concept of Roman domination can be formulated in terms of graphs. We consider only simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u: uv \in E(G)\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A *Roman dominating function (RDF)* of a graph G is a function $f: V(G) \to \{0, 1, 2\}$ such that whenever $f(v) = 0$ there exists a vertex $u \in N(v)$ with $f(u) = 2$. The *weight* of f is $\sum_{v\in V(G)} f(v)$. The *Roman domination number* $\gamma_R(G)$ of G is the minimum weight of RDFs of G.

Cockayne et al. [3] showed that $\gamma(G) \le \gamma_R(G) \le 2\gamma(G)$ and that $\gamma(G) = \gamma_R(G)$ implies G has no edges, where $\gamma(G)$ is the domination number of G. They also established that $\gamma_R(G) \le \frac{2n}{\delta(G)+1}\left(\ln \frac{\delta(G)+1}{2} + 1\right)$ for a graph G of n vertices and minimum degree $\delta(G)$. Chambers et al. [1] obtained that $\gamma_R(G) \le 4n/5$ when $\delta(G) \ge 1$ and $\gamma_R(G) \le 8n/11$ when $\delta(G) \ge 2$. The authors [7] proved that $\gamma_R(G) \le \max\{\lceil 2n/3\rceil, 23n/34\}$ for every 2-connected graph G. A characterization for graphs G with $\gamma_R(G) = \gamma(G) + k$ for $1 \le k \le \gamma(G)$ was given in [3,5,10]. Algorithmic results on Roman domination were given by Liedloff et al. [6]. Applications of Roman domination was also shown in [1].

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Among the results, we are most interested in those given in [1] for the upper bounds on graphs with minimum degree at least one or two. In Section 2, we prove that $\gamma_R(G) \le 2n/3$ for graphs G of n vertices with minimum degree at least three. We prove that this bound is sharp in Section 3. Haynes et al. [4] gave that the domination number of a connected, claw-free and net-free graph of n vertices is bounded by $\lceil n/3 \rceil$. In Section 4, we prove an analogous result that the Roman domination number of a connected, big-claw-free and big-net-free graph of n vertices is bounded by $\lceil 2n/3 \rceil$. Consequently, $\gamma_R(G) \le \lceil 2n/3 \rceil$ for all connected cocomparability graphs. We also give graphs to show that both big-claw-freeness and big-net-freeness are necessary to admit the upper bound.

2. Upper bound for graphs G with $\delta(G) \ge 3$

This section establishes the first main result as follows.

Theorem 1. *If G is a graph of n vertices and $\delta(G) \ge 3$, then $\gamma_R(G) \le 2n/3$.*

To prove the theorem, we employ the method introduced in [7]. For technical reasons, we consider three RDFs f_1, f_2 and f_3 at the same time. We use $\vec{f}$ to denote the 3-tuple (f_1, f_2, f_3), and $\vec{f}(v)$ for $(f_1(v), f_2(v), f_3(v))$. The *weight* of $\vec{f}$ is $w(\vec{f}) = \sum_{j=1}^{3} w(f_j)$. Notice that $w(f_j) \le w(\vec{f})/3$ for some j. For any subgraph H of G, let $w(\vec{f}, H)$ be $\sum_{j=1}^{3} \sum_{v \in V(H)} f_j(v)$. A vertex v is $\vec{f}$*-strong* if $f_j(v) = 2$ for some j. The following useful lemmas are from [7].

Lemma 2 ([7]). *If n is a multiple of 3, then the n-cycle C_n has a 3-tuple $\vec{f}$ of RDFs in which all vertices are $\vec{f}$-strong and $w(\vec{f}) \le 2n$.*

Lemma 3 ([7]). *Suppose G has a 3-tuple $\vec{f}$ of RDFs for which u and v are $\vec{f}$-strong. If G' is obtained from G by adding a disjoint path $P = v_1v_2 \dots v_t$ with $t \ge 1$ and two edges uv_1 and v_tv, then $\vec{f}$ can be extended to G' such that $w(\vec{f}, P) = 2t$ and v_i is $\vec{f}$-strong for $1 < i < t$.*

A *tailed* $(t-s+1)$*-cycle* H is a path $v_1v_2 \dots v_t$ together with an edge v_tv_s, denoted by $v_1v_2 \dots v_t + v_tv_s$, where $1 \le s \le t-2$. In this definition, we call vertex v_1 is the *starting vertex*, path $v_1, v_2, \dots, v_s$ the *tail*, cycle $v_sv_{s+1} \dots v_tv_s$ of length $t-s+1$ the *body* and vertex v_t the *inner vertex* of H. As pointed out by a referee, a tailed cycle has been called a lasso.

Lemma 4 ([7]). *Suppose G has a 3-tuple $\vec{f}$ of RDFs for which u is $\vec{f}$-strong. If G' is obtained from G by adding a tailed t'-cycle H with $t' \equiv 1 \pmod 3$ and an edge uv_1, then $\vec{f}$ can be extended to G' such that $w(\vec{f}, H) = 2|V(H)|$ and all vertices of H except the inner vertex are $\vec{f}$-strong.*

For a subgraph L of G, a vertex in L is a *boundary* (resp. *interior*) vertex if it is adjacent to some (resp. no) vertex in $V(G) - V(L)$.

Lemma 5. *Suppose G is a connected graph with $\delta(G) \ge 3$, and L is a non-null subgraph of G. If $V(L) \ne V(G)$ and $G' = G - V(L)$ has no $3p$-cycles as subgraphs, then* (i) *or* (ii) *holds.*

(i) *G' has a path P whose end vertices are adjacent to vertices in L and are interior vertices of $L \cup P$.*
(ii) *G' has a tailed $(3p+1)$-cycle H whose starting vertex is adjacent to a vertex in L and whose inner vertex is an interior vertex of $L \cup H$.*

Proof. Let $\mathcal{P}$ be the set of all longest paths $P = v_1v_2 \dots v_t$ in G' with v_1 adjacent to some vertex in L, and let I be the set of all v_t in such a path P. Since G is connected, $\mathcal{P}$ is non-empty. Since $P \in \mathcal{P}$ is longest, the vertex $v_t \in I$ corresponding to P is an interior vertex of $L \cup P$. If some vertex $v_t \in I$ is adjacent to a vertex in L, then the path $P^{-1} = v_tv_{t-1} \dots v_1 \in \mathcal{P}$ and $v_1 \in I$. This gives that v_1 and v_t are interior vertices of $L \cup P$ and so (i) holds. Hence, we have the following.

Claim 1. All neighbors of any vertex $v_t \in I$ corresponding to $P \in \mathcal{P}$ are in $V(P)$.

Choose a path $P = v_1v_2 \dots v_t$ in $\mathcal{P}$. We show that (ii) holds unless $i \equiv t-1 \pmod 3$ and $v_{i+1} \in I$ whenever $1 \le i \le t-2$ and v_tv_i is an edge. Consider the tailed $(t-i+1)$-cycle $H = v_1v_2 \dots v_t + v_tv_i$. Since v_1 is adjacent to some vertex in L and the neighbors of v_t are all in H by Claim 1, (ii) is true unless $t-i+1 \not\equiv 1 \pmod 3$. This together with the assumption that $t-i+1 \not\equiv 0 \pmod 3$ implies that $i \equiv t-1 \pmod 3$. Finally, $v_{i+1} \in I$ follows from the fact that $P_i = v_1v_2 \dots v_{i-1}v_iv_tv_{t-1} \dots v_{i+2}v_{i+1}$ is in $\mathcal{P}$. So we have the following.

Claim 2. If $1 \le i \le t-2$ and v_tv_i is an edge, then $i \equiv t-1 \pmod 3$ and $v_{i+1} \in I$.

Since $\delta(G) \ge 3$, by Claim 1, vertex v_t is adjacent to two vertices $v_{t'}$ and v_s in P with $1 \le s < t' \le t-2$. According to Claim 2, $s \equiv t' \equiv t-1 \pmod 3$ and $v_{t'+1} \in I$ since $P_{t'} = v_1v_2 \dots v_{t'-1}v_{t'}v_tv_{t-1} \dots v_{t'+2}v_{t'+1} \in \mathcal{P}$. We may assume that t' is chosen as large as possible and s is the second largest. Furthermore, we assume that $t-t'$ is as small as possible among all paths in $\mathcal{P}$.

Consider the path $P_{t'} \in \mathcal{P}$ in which $v_{t'+1} \in I$ is adjacent to $v_{t'+2}$ and $v_{t'}$. Similarly, by $\delta(G) \ge 3$ and Claim 1, vertex $v_{t'+1}$ has a neighbor $v_{s'}$ other than $v_{t'+2}$ and $v_{t'}$ in $V(P_{t'}) = V(P)$. By the minimality of $t-t'$, we have $s' < t'$. By Claim 2, $s' \equiv t' \equiv t-1 \pmod 3$. Hence $s \equiv s' \pmod 3$. We may assume that under this condition, s' is chosen as large as possible.

Then, by the second maximality of s, if v_t has any neighbor v_i other than v_{t-1}, $v_{t'}$ and v_s, then $i < s$. Similarly, if $v_{t'+1}$ has any neighbor v_j other than $v_{t'+2}$, $v_{t'}$ and $v_{s'}$, then $j < s'$.

If $s \geq s'$, then consider the tailed $(s - s' + 4)$-cycle $H = v_1v_2 \dots v_{s'} \dots v_s v_t v_{t'} v_{t'+1} + v_{t'+1}v_{s'}$ whose inner vertex $v_{t'+1}$ can only be adjacent to vertices in $\{v_1, v_2, \dots, v_{s'-1}\}$. If $s < s'$, then consider the tailed $(s' - s + 4)$-cycle $H = v_1v_2 \dots v_s v_t v_{t'} v_{t'+1} v_{s'} v_{s'-1} \dots v_s$ whose inner vertex v_t can only be adjacent to vertices in $\{v_1, v_2, \dots, v_{s-1}\}$. In either case, (ii) holds. □

The following result is useful to our proof of Theorem 1.

Theorem 6 ([2]). *If G is a graph of order n with at most one vertex of degree less than three, then G contains a cycle of length $3p$.*

We are now ready to prove the main theorem.

Proof of Theorem 1. Without loss of generality, we may assume that G is connected. Let $\mathcal{C}$ be a maximal collection of vertex-disjoint cycles of length multiples of 3 in G, and L be the union of graphs in $\mathcal{C}$. We note that L is non-empty by Theorem 6. By Lemma 2, there is a 3-tuple of RDFs $\vec{f}$ of L in which all boundary vertices of L are $\vec{f}$-strong and $w(\vec{f}, L) \leq 2\ell$, where ℓ is the number of vertices of L. If $V(L) \neq V(G)$, we can always find a subgraph H on h vertices from $G - V(L)$ satisfying one of the two cases by Lemma 5. We then obtain a new graph L' by adding H and some edges between H and L to L to get a new graph L' on $\ell' = \ell + h$ vertices. Then use Lemmas 3 and 4, respectively, to extend $\vec{f}$ to a 3-tuple of RDF of L' such that all boundary vertices of L' are $\vec{f}$-strong and $w(\vec{f}, H) \leq 2h$, or equivalently $w(\vec{f}, L') \leq 2\ell'$. We then replace L by L' and continue the process until $\ell = n$. □

3. Lower bound for 3-regular 3-connected graphs

In this section, we show that the upper bound on Roman domination numbers of graphs with minimum degree at least 3 in Theorem 1 is sharp. An *independent set* is a set of pairwise non-adjacent vertices. For any RDF f and $0 \leq i \leq 2$, $f^{-1}(i)$ denotes the set of vertices x with $f(x) = i$.

Lemma 7. *If G is a graph with $|N(x) \setminus N(y)| \leq 2$ for all $xy \in E(G)$, then G has an RDF f with $w(f) = \gamma_R(G)$ and $f^{-1}(1) \cup f^{-1}(2)$ is an independent set.*

Proof. Choose an RDF f with $w(f) = \gamma_R(G)$. First, there is no edge xy with $f(x) = 1$ and $f(y) = 2$ for otherwise changing $f(x)$ to 0 gives an RDF f' with $w(f') < w(f)$, a contradiction.

Next, suppose $f(x) = f(y) = 2$ for some edge xy. Then, we derive a new RDF f' by changing $f(y)$ to 0 and $f(z)$ to 1 for all vertices $z \in N(y) \setminus N(x)$ with $f(z) = 0$. By the assumption, there are at most 2 such z and so $w(f') \leq w(f)$. Repeating this process shows that $f^{-1}(2)$ is independent.

Finally, suppose $f(x) = f(y) = 1$ for some edge xy. Then, changing $f(x)$ to 2 and $f(y)$ to 0 gives a new RDF f' with $w(f') \leq w(f)$. Repeat this process, we may assume that $f^{-1}(1)$ is independent. Notice that this process keeps $f^{-1}(2)$ independent, since x has no neighbor z with $f(z) = 2$. □

The hypothesis of Lemma 7 holds for graphs with maximum degree at most 3.

Theorem 8. *There are infinitely many cubic 3-connected graphs G of order n with $\gamma_R(G) \geq 2n/3$.*

Proof. For any integer $t \geq 3$, we construct graph G_t from the union of two disjoint $3t$-cycles $x_1, x_2, \dots, x_{3t}, x_1$ and $y_1, y_2, \dots, y_{3t}, y_1$ by adding edges $x_i y_{j_i}$ for $1 \leq i \leq 3t$, where $j_i = i$ if $i \equiv 1 \pmod 3$, $j_i = i + 1$ if $i \equiv 2 \pmod 3$ and $j_i = i - 1$ if $i \equiv 0 \pmod 3$; see Fig. 1. Notice that G_t is a cubic 3-connected graph on $n = 6t$ vertices.

Choose an RDF f of G_t with $w(f) = \gamma_R(G_t)$. Since G_t is cubic, we may assume that $f^{-1}(1) \cup f^{-1}(2)$ is independent by Lemma 7. Since $N[x_{3i}] \setminus N[y_{3i}] \subseteq N[x_{3i+1}]$, whenever $f(x_{3i}) = f(y_{3i}) = 2$ for some i, we may define a new RDF f' with $w(f') \leq w(f)$ by changing $f(x_{3i})$ to 1 when $f(x_{3i+2}) = 2$ and changing $(f(x_{3i}), f(x_{3i+1}), f(x_{3i+2}))$ to $(0, 2, 0)$ otherwise. Hence, we may assume that $f(x_{3i})$ and $f(y_{3i})$ cannot be both 2 for $1 \leq i \leq t$.

In the following, we shall show that $w(f) \geq 2n/3 = 4t$. Suppose to the contrary that $w(f) < 4t$. Let $A_i = \{x_{3i-2}, y_{3i-2}, x_{3i-1}, y_{3i-1}, x_{3i}, y_{3i}\}$ for $1 \leq i \leq t$. Since $(A_1, A_2, \dots, A_t)$ is a partition of $V(G_t)$, there is some i, say $i = 3$, such that $w(f, A_3) \leq 3$ but $w(f, A_2) \leq 4$. If $f(x_7) = 2$, then $f(y_8) = 1$ and $f(v) = 0$ for all other $v \in A_3$. This forces $f(x_{10}) = f(y_{10}) = 2$, where x_{10} is x_1 and y_{10} is y_1 when $t = 3$, to dominate x_9 and y_9, a contradiction to that $f^{-1}(2)$ is independent. Hence, $f(x_7) \neq 2$. By symmetry, $f(y_7) \neq 2$.

If $f(x_8) = 2$, then $f(y_8) = 1$ and $f(v) = 0$ for all other $v \in A_3$. This forces $f(y_6) = 2$ to dominate y_7. Hence, $f(x_5) = f(y_5) = 0$ and $f(x_6) = 1$ by the assumption that $f^{-1}(1) \cup f^{-1}(2)$ is independent and the fact that $f(x_6)$ and $f(y_6)$ cannot be both 2. Consequently, $f(x_4) + f(y_4) \leq 1$ as $w(f, A_2) \leq 4$. We may assume $f(x_4) = 0$ and $f(y_4) \leq 1$. Hence, $f(x_3) = 2$ and so $f(y_3) \leq 1$. This implies $f(y_4) = 1$ and which in turn implies $f(y_3) = 0$. But then y_3 is not dominated by any neighbor as $f(x_2) = f(y_2) = 0$ due to $f(x_3) = 2$, a contradiction. Hence, $f(x_8) \neq 2$. By symmetry, $f(y_8) \neq 2$.

If $f(x_9) = 2$, then $f(x_8) = f(y_8) = 0$ and $f(x_7) + f(y_7) \leq 1$, since $w(f, A_3) \leq 3$. We may assume that $f(x_7) = 0$ and $f(y_7) \leq 1$. Hence, $f(x_6) = 2$ and so $f(y_6) \leq 1$. This implies $f(y_7) = 1$ and which in turn implies $f(y_6) = 0$. But then y_6 is

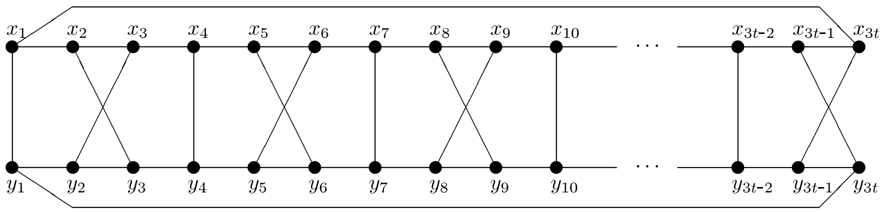

Fig. 1. Graph G_t.

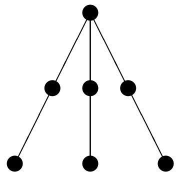

Fig. 2. A big-claw.

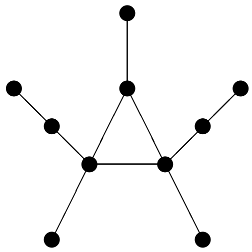

Fig. 3. A big-net.

not dominated by any neighbor as $f(x_5) = f(y_5) = 0$ due to $f(x_6) = 2$, a contradiction. Hence, $f(x_9) \neq 2$. By symmetry, $f(y_9) \neq 2$.

In summary, $f(v) \leq 1$ for all $v \in A_3$. Then $f(x_8) = f(y_8) = 1$. Since either $f(x_{10}) \neq 2$ or $f(y_{10}) \neq 2$, where x_{10} is x_1 and y_{10} is y_1 when $t = 3$, $f(x_9) + f(y_9) \geq 1$, a contradiction to that $f^{-1}(1)$ is independent. Hence, $\gamma_R(G_t) = w(f) \geq 4t$ as desired. □

4. Big-claw-free and big-net-free graphs

A graph is *H-free* if it does not contain H as an induced subgraph. Haynes et al. [4] proved that $\gamma(G) \leq \lceil n/3 \rceil$ for any connected, claw-free and net-free graph G of n vertices. The aim of this section is to establish an analogous result for Roman domination. More precisely, we show that $\gamma_R(G) \leq \lceil 2n/3 \rceil$ for any connected, big-claw-free and big-net-free graph G of n vertices.

For nonnegative integers s and t, an (s, t)-*star* is the graph obtained from identifying s copies of P_2's and t copies of P_3's at one leaf of each graph, which is called the *center* of the (s, t)-star. Notice that an $(s, 0)$-star is the star $K_{1,s}$ and a $(3, 0)$-star is a claw. A *big-claw* is a $(0, 3)$-star; see Fig. 2. Claw-free graphs are well studied with fruitful results. For our concern of Roman domination, we deal with big-claw-free graphs which include claw-free graphs.

For the result on Roman domination, we need the graph *big-net* as in Fig. 3.

Lemma 9. *If G is an (s, t)-star of order n, then $\gamma_R(G) \leq \lceil 2n/3 \rceil$. Furthermore, if $2s + t \geq 4$ or $(s, t) = (0, 1)$, then $\gamma_R(G) \leq 2n/3$.*

Proof. Since the $(0, 1)$-star is isomorphic to the $(2, 0)$-star, we may assume that $(s, t) \neq (0, 1)$ and so $2s + t \geq 2$. Suppose x is the center of G. Define an RDF f of G by $f(x) = 2$, $f(y) = 0$ for $y \in N(x)$ and $f(z) = 1$ for $z \notin N[x]$. Then $w(f) = 2 + t$. Since $n = 1 + s + 2t$ and $2s + t \geq 2$, we have $w(f) = 2 + t \leq (2n + 2)/3$ and so $\gamma_R(G) \leq \lceil 2n/3 \rceil$. For the case of $2s + t \geq 4$, we have $\gamma_R(G) \leq w(f) \leq 2n/3$. □

Theorem 10. *If G is a connected big-claw-free and big-net-free graph of n vertices, then $\gamma_R(G) \leq \lceil 2n/3 \rceil$.*

Proof. The theorem is clear for $n \leq 3$. Suppose to the contrary that the theorem is not true. Choose a minimum counter-example G to the theorem, that is, $\gamma_R(G) > \lceil 2n/3 \rceil$ but $\gamma_R(H) \leq \lceil 2n'/3 \rceil$ for any proper connected induced subgraph H of G of n' vertices.

Apply a depth-first-search (DFS) to G to produce a tree T rooted at an arbitrary vertex r. Notice that every edge in G is either a tree edge or a back edge but not a cross edge between two non-comparable vertices, i.e. none is an ancestor of the other. For any vertex x, let C_x be the set of all children of x in T, D_x the set of all descendants of x in T, T_x the subtree of T induced by D_x and G_x the subgraph of G induced by D_x. The *height* $h(x)$ of a vertex x is the largest length of a path from x to a vertex in D_x in T. Notice that if p is the parent of x, then $h(p) \geq 1 + h(x)$.

In the following, we will frequently use the fact that if $S \subseteq V(G)$ induces a subgraph $G[S]$ with $\gamma_R(G[S]) \leq 2|S|/3$, then $\gamma_R(G) \leq \gamma_R(G-S) + \gamma_R(G[S]) \leq \lceil 2(n-|S|)/3 \rceil + 2|S|/3 \leq \lceil 2n/3 \rceil$ which gives a contradiction. For instance, if there is a vertex x_1 with $h(x_1) = 1$ having at least two children, then $\gamma_R(G_{x_1}) \leq 2 \leq 2|D_{x_1}|/3$.

Suppose that x_2 is a vertex with $h(x_2) = 2$ having s children of height 0 and t children of height 1. We shall prove that $s = t = 1$ and leaves in T_{x_2} are also of degree 1 in G. Notice that each child of x_2 of height 1 has exactly one child, so $|D_{x_2}| = 1 + s + 2t$. For the case when x_2 is adjacent to a grandchild, G_{x_2} contains an $(s+2, t-1)$-star. By Lemma 9, $\gamma_R(G_{x_2}) \leq 2|D_{x_2}|/3$, a contradiction. Since the DFS creates no cross edges, we now may assume that $G_{x_2} = T_{x_2}$ is an (s, t)-star. Again, by Lemma 9, we may assume that $2s + t \leq 3$ and $(s, t) \neq (0, 1)$. Since $h(x_2) = 2$ and G is big-claw-free, $1 \leq t \leq 2$. Hence, (s, t) can only be $(0, 2)$ or $(1, 1)$.

Suppose $(s, t) = (0, 2)$. If any vertex in $D_{x_2} - \{x_2\}$ has a back edge to a proper ancestor of x_2, then x_2 together with a child and its child form a set S such that $G - S$ is connected and $\gamma_R(G[S]) \leq 2$, a contradiction. So, the neighbors of any vertex in $D_{x_2} - \{x_2\}$ are all in D_x. Then the parent p of x_2 in T cannot have a child other than x_2, for otherwise T_p contains a big-claw with center x_2 because there are no cross edges. In this case, T_p is a $(1, 2)$-star and so $\gamma_R(G_p) \leq 2|D_p|/3$, a contradiction. This proves that $s = t = 1$. In other words, x_2 has a child x'_0 of height 0 and a child x_1 of height 1 with a child x_0 of height 0, and x_0 and x'_0 are the only leaves of T_{x_2}. Furthermore, x_0 or x'_0 has degree 1 in G since the neighbors of any vertex in $D_{x_2} - \{x_2\}$ are all in D_x. This proves the following claim.

Claim 1. If $h(x_2) = 2$, than x_2 has exactly two children x_1 and x'_0, where x'_0 is of degree 1 in G and x_1 has exactly one child x_0 of degree 1 in G.

For any vertex x_3 of height 3, we define:

$$U_{x_3,0} = \{x_0 \in D_{x_3} \cap N(x_3) \colon h(x_0) = 0 \text{ or } h(x_0) = 1 \text{ but its child is adjacent to } x_3\},$$
$$U_{x_3,1} = \{x_1 \in C_{x_3} \colon h(x_1) = 1 \text{ and its child is not adjacent to } x_3\},$$
$$U_{x_3,2} = \{x_2 \in C_{x_3} \colon h(x_2) = 2\},$$
$$U'_{x_3,1} = \{x_1 \in U_{x_3,1} \colon \text{ no vertex in } D_{x_1} \text{ has a back edge to a proper ancestor of } x_3\},$$
$$U'_{x_3,2} = \{x_2 \in U_{x_3,2} \colon \text{ no vertex in } D_{x_2} \text{ has a back edge to a proper ancestor of } x_3\}.$$

Notice that $U_{x_3,2} \neq \emptyset$ by the definition of the height function. By Claim 1 and the definition, each vertex in $U_{x_3,2} \cup U_{x_3,1}$ has a child not adjacent to x_3. Since there is no cross edge and G is big-claw-free, $|U_{x_3,2} \cup U_{x_3,1}| \leq 2$. Consequently, $|U_{x_3,1}| \leq 1$.

Now, we shall prove that $U_{x_3,0}$ is nonempty. Suppose to the contrary that $U_{x_3,0} = \emptyset$, then we have the following four cases.

- If $|U_{x_3,2}| = 2$, then $|D_{x_3}| = 9$ and $\gamma(D_{x_3}) \leq 6 = 2|D_{x_3}|/3$, a contradiction.
- If $|U_{x_3,2}| \leq 1$ and $U'_{x_3,1} = \emptyset$. Choose $x_2 \in U'_{x_3,2}$ if $U'_{x_3,2} \neq \emptyset$, and $x_2 \in U_{x_3,2}$ for otherwise. Then $G - S$ is connected and $\gamma_R(G[S]) \leq 2|S|/3$ as $G[S]$ is a $(2, 1)$-star, where $S = \{x_3\} \cup D_{x_2}$, a contradiction.
- If $|U'_{x_3,2}| = |U'_{x_3,1}| = 1$, then $|D_{x_3}| = 7$. Notice that $G - D_{x_3}$ is connected, and every vertex in $D_{x_3} - \{x_3\}$ is not adjacent to any vertex in $V(G) - D_{x_3}$. Then all vertices in $V(G) - D_{x_3}$ are adjacent to x_3, for otherwise x_3 is adjacent to some vertex u but not adjacent to some vertex v in $N(u) - D_{x_3}$, where $N(v) \cap D_{x_3} = \emptyset$. So $G[D_{x_3} \cup \{u, v\}]$ contains an induced big-claw, a contradiction. It is then easy to see that $\gamma_R(G) \leq 5 \leq \lceil 2n/3 \rceil$ when $V(G) = D_{x_3}$, and $\gamma_R(G) \leq 6 \leq \lceil 2n/3 \rceil$ when $V(G) \neq D_{x_3}$.
- If $U'_{x_3,2} = \emptyset$ and $U'_{x_3,1} = \{x_1\}$, then $G - S$ is connected and $\gamma_R(G[S]) \leq 2 = 2|S|/3$, where $S = \{x_3\} \cup D_{x_1}$, a contradiction.

Let $x''_0 \in U_{x_3,0}$. Now, we shall prove that $U'_{x_3,2}$ is empty. Suppose to the contrary that $x_2 \in U'_{x_3,2}$. By Claim 1, x_2 has a child x'_0 of degree 1 in G and another child x_1 having exactly one child x_0 that is of degree 1 in G. Notice that $G' = G - \{x_0, x_1, x''_0\}$ is connected. By the minimality of G, G' has an RDF f' of weight $w(f') \leq \lceil 2(n-3)/3 \rceil$.

For the case when $f'(x_3) = 2$, we define a function f on $V(G)$ by $f(v) = f'(v)$ for $v \in V(G'), f(x_1) = 2$ and $f(x_0) = f(x''_0) = 0$. Then f is an RDF of G and so $\gamma_R(G) \leq w(f) = w(f') + 2 \leq \lceil 2(n-3)/3 \rceil + 2 = \lceil 2n/3 \rceil$, a contradiction.

For the case when $f'(x_3) \leq 1$, since x'_0 only adjacent to x_2 and x_2 only adjacent to x'_0 and x_3 in G', $f'(x'_0) + f'(x_2) + f'(x_3) \geq 2$ and $f'(x'_0) + f'(x_2) + f'(x_3) \geq 3$ if $f'(x_2) = 2$. When $f'(x_3) = 2$, we define a function f on $V(G)$ by $f(v) = f'(v)$ for $v \in V(G') - \{x'_0, x_2\}, f(x_2) = 2, f(x_0) = 1$ and $f(x'_0) = f(x_1) = f(x_3) = 0$; when $f'(x_3) \neq 2$, we define a function f on $V(G)$ by $f(v) = f'(v)$ for $v \in V(G') - \{x'_0, x_2, x_3\}, f(x_2) = 2, f(x_0) = f(x''_0) = 1$ and $f(x'_0) = f(x_1) = f(x_3) = 0$. Then f is an RDF of G and so $\gamma_R(G) \leq w(f) \leq w(f') + 2 \leq \lceil 2(n-3)/3 \rceil + 2 = \lceil 2n/3 \rceil$, a contradiction. This proves that $U'_{x_3,2}$ is empty.

Consequently, let $s = |U_{x_3,0}|$ and $t = |U_{x_3,1}|$, then $S = \{x_3\} \cup U_{x_3,0} \cup (\bigcup_{x_1 \in U_{x_3,1}} D_{x_1})$ induces an (s, t)-star such that $G - S$ is connected. If $2s + t \geq 4$ or $(s, t) = (0, 1)$, then $\gamma_R(G[S]) \leq 2|S|/3$ by Lemma 9, a contradiction. So $(s, t) = (1, 0)$ or $(1, 1)$. Let x_2 be an arbitrary vertex in $U_{x_3,2}$.

For the case of $(s, t) = (1, 0)$, the set $S' = \{x_3\} \cup U_{x_3,0} \cup D_{x_2}$ induces a $(1, 2)$-star and $G - S'$ is connected since $U'_{x_3,2}$ is empty. By Lemma 9, $\gamma(G[S']) \leq 2|S'|/3$, a contradiction. For the case of $(s, t) = (1, 1)$, say $U_{x_3,0} = \{x''_0\}$ and $U_{x_3,1} = \{x'_1\}$. By

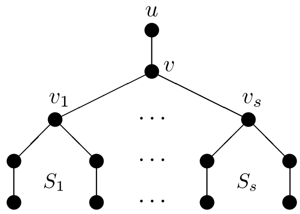

Fig. 4. A big-net-free but not big-claw-free graph.

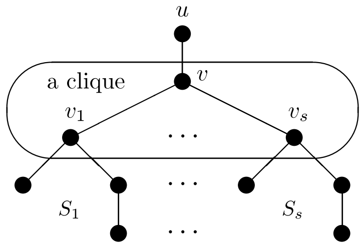

Fig. 5. A big-claw-free but not big-net-free graph.

Claim 1, x_2 has a child x_0' of degree 1 in G and a child x_1 who has a child x_0 of degree 1 in G. If x_1 is adjacent to x_3, then D_{x_3} has a spanning $(1, 3)$-star such that $G - D_{x_3}$ is connected and $\gamma_R(G_{x_3}) \le 2|D_{x_3}|/3$, a contradiction. If x_1 has a back edge to a proper ancestor of x_3, then $G[S'']$ has a spanning $(1, 2)$-star, where $S'' = D_{x_3} - \{x_1, x_0\}$, such that $G - S''$ is connected and $\gamma_R(G[S'']) \le 2|S''|/3$, a contradiction. Similarly, if any vertex in $D_{x_1'}$ has a back edge to a proper ancestor of x_3, then $G[S''']$ has a spanning $(1, 2)$-star, where $S''' = D_{x_3} - D_{x_1'}$, such that $G - S'''$ is connected, a contradiction. If x_0'' has a back edge to a proper ancestor of x_3, then $\{x_3\} \cup D_{x_1'}$ induced $(2, 0)$-star and $G - (\{x_3\} \cup D_{x_1'})$ is connected, a contradiction. Hence, together with the fact that $U'_{x_3,2}$ is empty, we may assume that x_2 is the only vertex in D_{x_3} having a back edge to a proper ancestor of x_3. Let p be the parent of x_3. If x_3 is the only child of p, then $|D_p| = 9$ and $\gamma_R(G_p) \le 6 = 2|D_p|/3$, which is a contradiction. So, p has another child p'. If p is adjacent to x_2, then $D_{x_3} \cup \{p, p'\}$ induces a big-net, otherwise it contains a big-claw; contradiction in either case. □

Since every cocomparability graph is big-claw-free and big-net-free, we have the following.

Corollary 11. *For any connected cocomparability graph G on n vertices, $\gamma_R(G) \le \lceil 2n/3 \rceil$.*

We close this section by showing that both the big-claw-freeness and the big-net-freeness are necessary in Theorem 10. We first consider the graph G obtained from the disjoint union of a 2-path uv and $s \ge 2$ copies of $(0, 2)$-stars S_i centered at v_i by joining v to all v_i; see Fig. 4. Notice that G is a big-net-free but not big-claw-free graph of $n = 5s + 2$ vertices. Suppose f is an RDF of G. It is easy to see that $\sum_{x \in V(S_i)} f(x) \ge 4$ for $1 \le i \le s$ and $f(u) + f(v) \ge 1$. Hence $w(f) \ge 4s + 1 > (2n + 2)/3$ and so $\gamma_R(G) > \lceil 2n/3 \rceil$.

Next, we consider the graph H obtained from the disjoint union of a 2-path uv and $s \ge 4$ copies of $(1, 1)$-stars S_i centered at v_i, and making $\{v, v_1, v_2, \ldots, v_s\}$ a clique; see Fig. 5. Notice that H is a big-claw-free but not big-net-free graph of $n = 4s + 2$ vertices. Suppose f is an RDF of G. It is easy to see that $\sum_{x \in V(S_i)} f(x) \ge 3$ for $1 \le i \le s$ and $f(u) + f(v) \ge 1$. Hence $w(f) = 3s + 1 > (2n + 2)/3$ and so $\gamma_R(G) > \lceil 2n/3 \rceil$.

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